A Volumetric Proof of the Log-Concavity of the Characteristic Polynomial of Matroids

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Conjecture (Heron–Rota–Welsh ['53-ish])

The coefficients of the characteristic polynomial of a matroid are log-concave.

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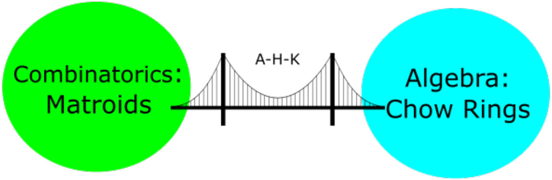
Theorem (Adiprasito–Huh–Katz [2018])

The Heron-Rota-Welsh conjecture is true.

How Will We Do This?



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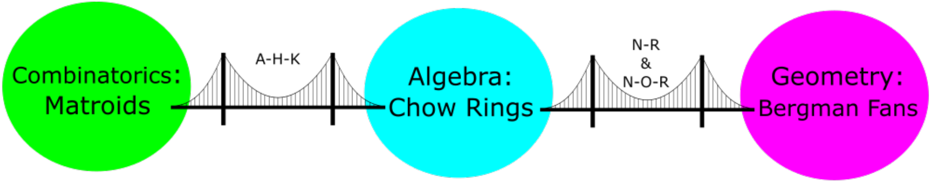

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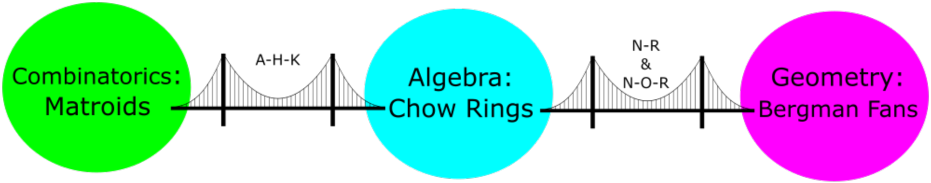
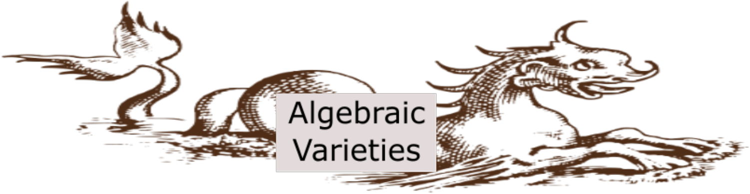
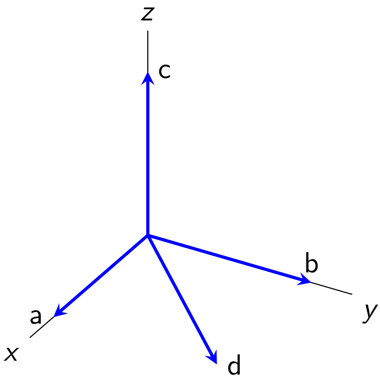


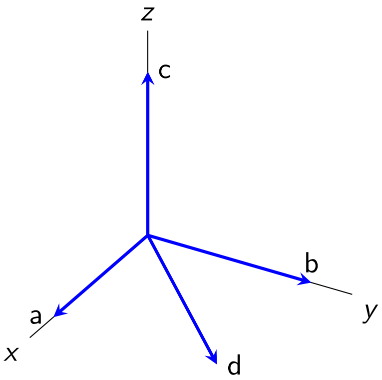


$$E = \{a, b, c, d\}$$



What is a Matroid?

$$E = \{a, b, c, d\}$$



$$\mathcal{I} = \{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, acd, bcd\}$$

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"A Generalization of the Notion of Independence"

Definition (Matroid — Independent Set Axioms)

A *matroid* is a 2-tuple $\mathcal{M} = (E, \mathcal{I})$, where E is a finite set, called the *ground set*, and $\mathcal{I} \subseteq 2^E$ is a collection of subsets of E, called the *independent sets*, with the following properties:

1. $\emptyset \in \mathcal{I}$.
2. If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.
3. If $I_1, I_2 \in \mathcal{I}$ and $|I_1| \leq |I_2|$, then there exists some $e \in I_2 \setminus I_1$ such that $I_1 \cup e \in \mathcal{I}$.

Definition (Rank)

Let $\mathcal{M}$ be a matroid with groud set E. The *rank* is the map

$$\mathrm{rk}_{\mathcal{M}} \,:\, 2^E \to \mathbb{Z}_{\geq 0}$$

that takes any $X \subseteq E$ to the size of the largest independent set contained in X.

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Definition (Closure)

Given a matroid $\mathcal{M}$ wth ground set E, the *closure* is a map

$$\mathrm{cl}_{\mathcal{M}} \ : \ 2^E \to 2^E$$
$$X \mapsto \{e \in E \mid \mathrm{rk}(X \cup e) = \mathrm{rk}(X)\}.$$

For any $X \subseteq E$, we call $\mathrm{cl}(X)$ the closure of X.

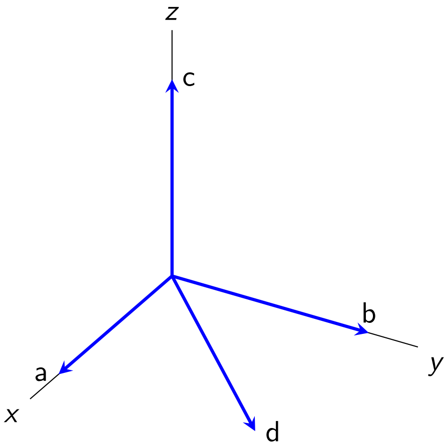
Definition (Flat)

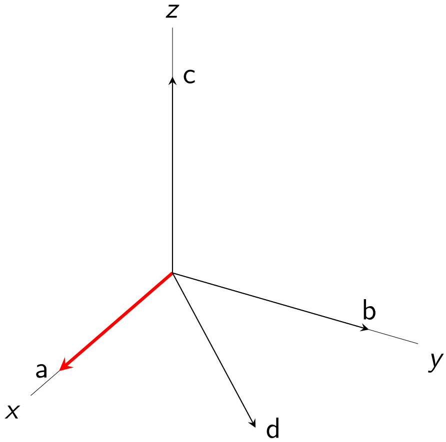
Given a matroid $\mathcal{M}$ and subset $X \subseteq E$, if

$$X = \text{cl}(X),$$

then X is a *flat* of $\mathcal{M}$.

Thinking About Some Flats

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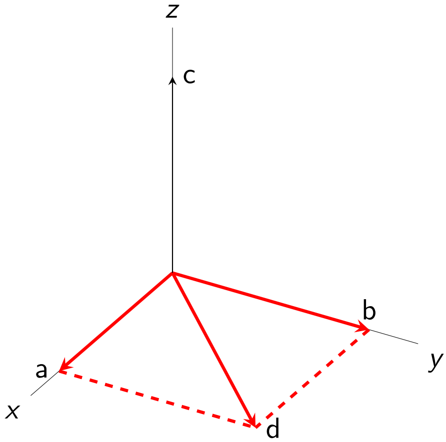
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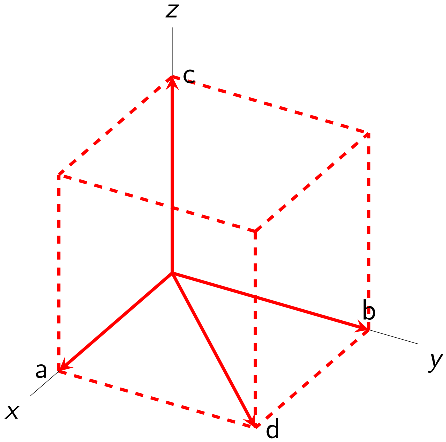
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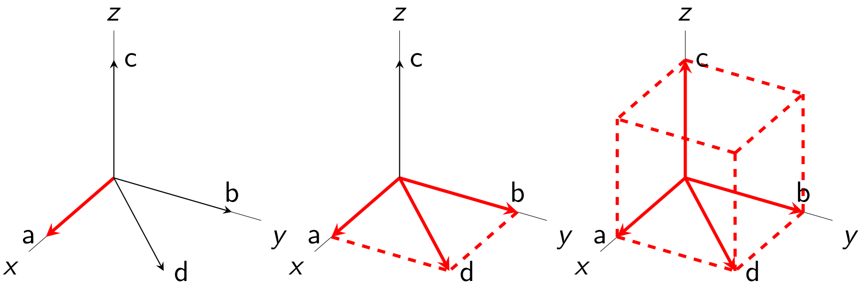

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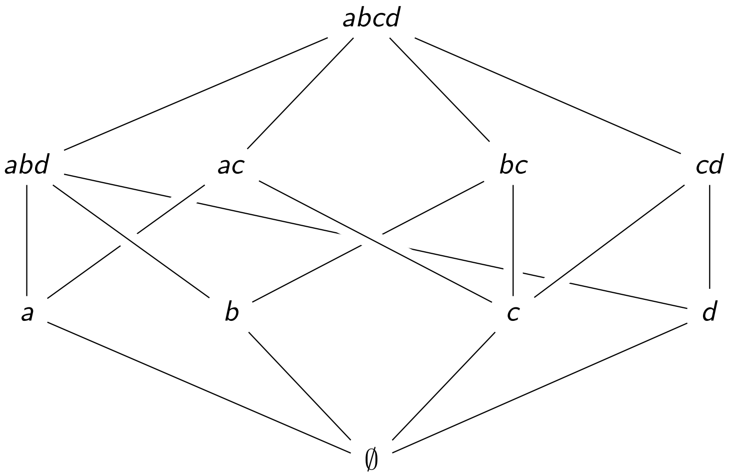
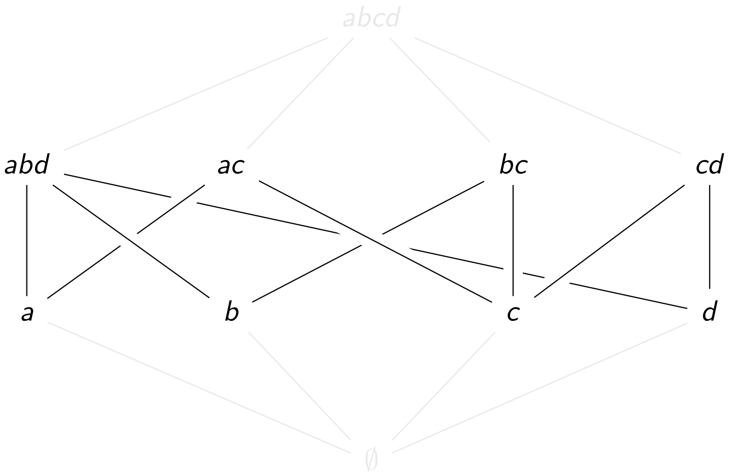


Proper Flats

$$\mathcal{L}^* = \{a, b, c, d, abd, ac, bc, cd\}$$



Definition (Characteristic Polynomial)

Let $\mathcal{M}$ be a matroid. Then the *characteristic polynomial* of $\mathcal{M}$ is given by

$$\chi_{\mathcal{M}}(z) = \sum_{X \subseteq E} (-1)^{|X|} z^{\mathrm{rk}(\mathcal{M}) - \mathrm{rk}(X)}.$$

Definition (Whitney Numbers of the First Kind)

Let $\mathcal{M}$ be a matroid with characteristic polynomial

$$\chi_{\mathcal{M}}(z) = \sum_{X \subseteq E} (-1)^{|X|} z^{\mathrm{rk}(\mathcal{M}) - \mathrm{rk}(X)}$$
$$= \sum_{k=0}^{\mathrm{rk}(\mathcal{M})} (-1)^k w_k z^{\mathrm{rk}(\mathcal{M}) - k}.$$

The unsigned portion of the coefficients, $w_0, w_1, \ldots, w_{\mathrm{rk}(\mathcal{M})}$, are the *Whitney numbers of the first kind*.

The Heron-Rota-Welsh Conjecture: Revisited

Conjecture (Heron-Rota-Welsh ['53-ish])

Let $\mathcal{M}$ be a matroid. If $w_0, w_1, \ldots, w_{\text{rk}(\mathcal{M})}$ are the Whitney numbers of the first kind, then

$$w_i^2 \geq w_{i-1} w_{i+1}$$

for $0 < i < \text{rk}(\mathcal{M})$. That is, the absolute values of the coefficients of the characteristic polynomial of $\mathcal{M}$ are log-concave.

Definition (Reduced Characteristic Polynomial)

Let $\mathcal{M}$ be a matroid of rank $r+1$. The *reduced characteristic polynomial* of $\mathcal{M}$ is

$$\overline{\chi}_{\mathcal{M}}(z) = \frac{\chi_{\mathcal{M}}(z)}{(z-1)} = \sum_{k=0}^{r} (-1)^k \overline{w}_k z^{r-k},$$

where $\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_r$ are the reduced coefficients of $\overline{\chi}_{\mathcal{M}}(z)$.

Changing the Goal Slightly

Lemma

If the reduced coefficients

$$\{\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_r\}$$

are a log-concave sequence, then the Whitney numbers of the first kind of $\mathcal{M}$

$$\{w_0, w_1, \ldots, w_{r+1}\},$$

are also log-concave.

Definition (The Chow Ring of a Matroid)

Let $\mathcal{M} = (E, \mathcal{L})$ be a matroid. Associate a polynomial ring with $\mathcal{M}$ given by

$$P_{\mathcal{M}} = \mathbb{R}[x_F \mid F \in \mathcal{L}^*],$$

and let

$$I_{\mathcal{M}} = \langle x_{F_1} x_{F_2} \mid F_1 \not\subseteq F_2 \text{ and } F_2 \not\subseteq F_1 \rangle ,$$

$$J_{\mathcal{M}} = \left\langle \sum_{e_1 \in F} x_F - \sum_{e_2 \in F} x_F \;\middle|\; e_1, e_2 \in E \right\rangle$$

be ideals of $P_{\mathcal{M}}$.
The *Chow ring of* $\mathcal{M}$ is given by the quotient

$$A^{\bullet}(\mathcal{M}) = \frac{P_{\mathcal{M}}}{I_{\mathcal{M}} + J_{\mathcal{M}}}.$$

Recall our example matroid with proper flats:

$$\mathcal{L}^* = \{a, b, c, d, abd, ac, bc, cd\};$$

An Example

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We generate the polynomial ring

$$P_{\mathcal{M}} = \mathbb{R}[x_a, x_b, x_c, x_d, x_{abd}, x_{ac}, x_{bc}, x_{cd}];$$

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Elements of the ideal $I_{\mathcal{M}}$ will be any multiple of a monomial containing variables corresponding to non-comparable flats, such as

$$x_a x_d \in I_{\mathcal{M}} \quad \text{and} \quad x_c x_{abd} \in I_{\mathcal{M}}.$$

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The ideal $J_{\mathcal{M}}$ in turn is generated from differences of sums of all variables that contain a particular ground element. For example,

$$\begin{aligned} \sum_{a \in F} x_F - \sum_{c \in F} x_F &= x_a + x_{abd} + x_{ac} - x_c - x_{ac} - x_{cd} \\ &= x_a + x_{abd} - x_c - x_{cd} \in J_{\mathcal{M}}. \end{aligned}$$

Definition (Degree Map)

Let $\mathcal{M}$ be a matroid of rank $r+1$. The *degree map* of $\mathcal{M}$ is a linear map

$$\deg : A^r(\mathcal{M}) \to \mathbb{Z}$$

such that for any totally ordered subset of flats $\{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r\}$ of $\mathcal{M}$,

$$\deg\left(\prod_{i=1}^{r} x_{F_i}\right) = 1.$$

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Proposition

The degree map is well-defined and unique.

Some Important Elements of the Chow Ring

Definition (Divisor)

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Definition (α and β)

Let $\mathcal{M}$ be a matroid with ground set E. For every element $e \in E$ we define the divisors

$$\alpha_e = \sum_{e \in F} x_F \quad \text{and} \quad \beta_e = \sum_{e \notin F} x_F .$$

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Proposition

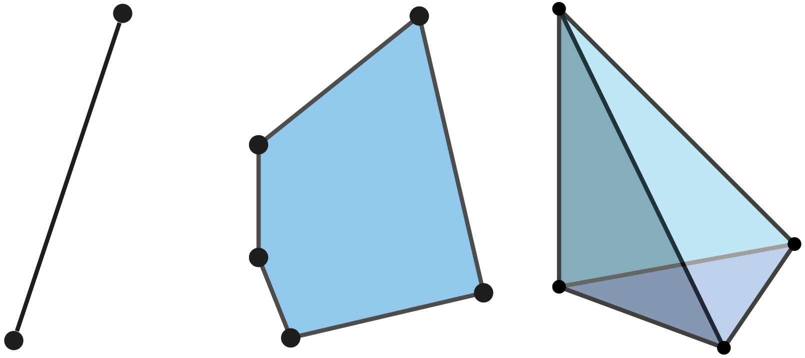
As elements of the Chow ring, $[\alpha_e]$ and $[\beta_e]$ are independent of the choice of $e \in E$. So we will write them just as α and β.

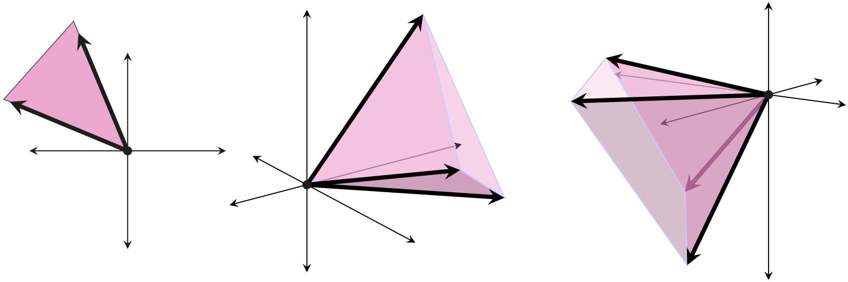
Lemma (A–H–K [2018])

Given any matroid $\mathcal{M}$ *with reduced coefficients* $\overline{w}_0, \ldots, \overline{w}_r$,

$$\overline{w}_k = \deg(\alpha^{r-k}\beta^k).$$

for all $0 \leq k \leq r$.



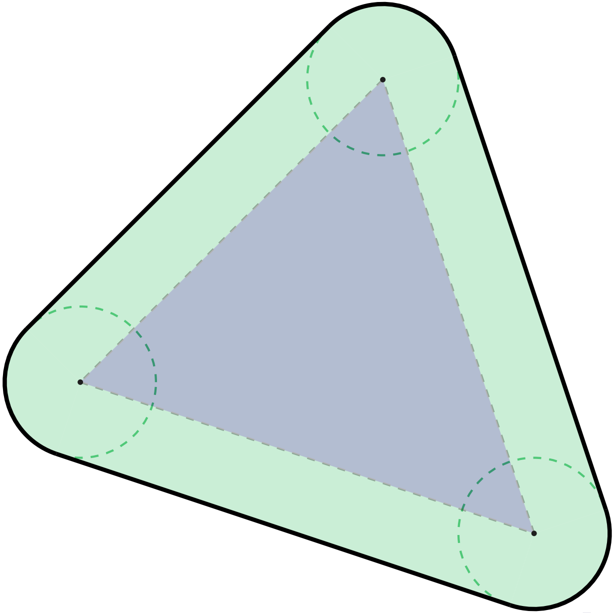


Definition (Minkowski Sum)

Let $P, Q \subseteq \mathbb{R}^n$. The *Minkowski sum* of P and Q is given by

$$P + Q = \{p + q \mid p \in P, q \in Q\}.$$

Minkowski Sum



Definition (Mixed Volume – Characterization)

The *mixed volume function* is a map MVol_n from ordered multisets $P_1, P_2, \ldots, P_n \subseteq \mathbb{R}^n$ of polytopes to $\mathbb{R}_{\geq 0}$, such that it has the following properties:

1. $\mathrm{MVol}_n(P, P, \ldots, P) = \mathrm{Vol}_n(P)$, for any polytope $P \subseteq \mathbb{R}^n$,
2. MVol_n is symmetric in all arguments, and
3. MVol_n is multilinear with respect to scaling and Minkowski addition.

Definition (Mixed Volume – As Coefficients)

Let $P_1, P_2, \ldots, P_\ell \subseteq \mathbb{R}^n$ be polytopes. The function

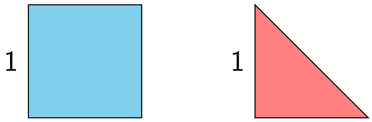
$$f(\lambda_1, \lambda_2, \ldots, \lambda_\ell) = \mathrm{Vol}(\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_\ell P_\ell), \quad \lambda_j \geq 0$$

is a homogeneous polynomial of degree n. It can be written symmetrically as

$$f(\lambda_1, \ldots, \lambda_\ell) = \sum_{j_1, j_2, \ldots, j_n = 1}^{\ell} \mathrm{MVol}(P_{j_1}, \ldots, P_{j_n}) \lambda_{j_1} \cdots \lambda_{j_n}.$$

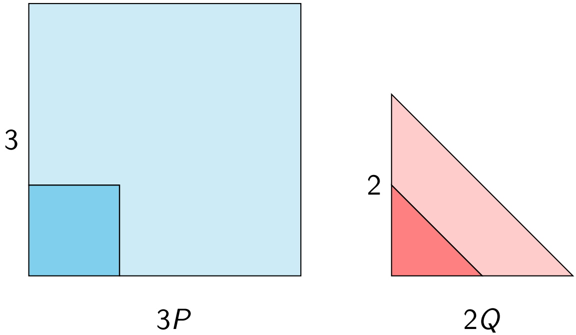
The coefficient associated to $\lambda_{j_1} \cdots \lambda_{j_n}$ is the *mixed volume* of $P_{j_1}, \ldots, P_{j_n}$.

Let's Do An Example



P Q

Let's Do An Example



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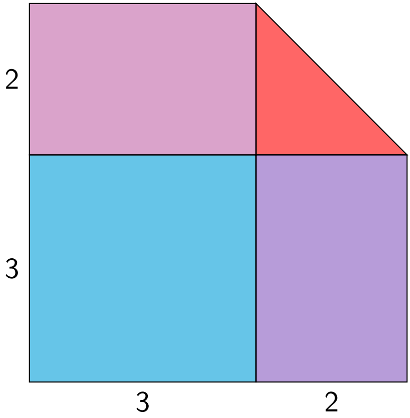
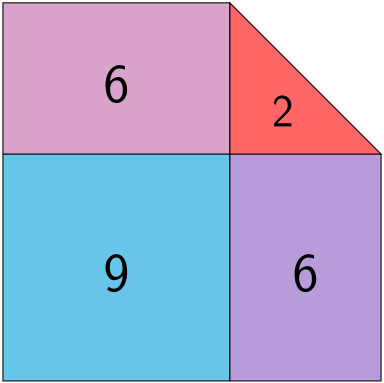


$$3P + 2Q$$



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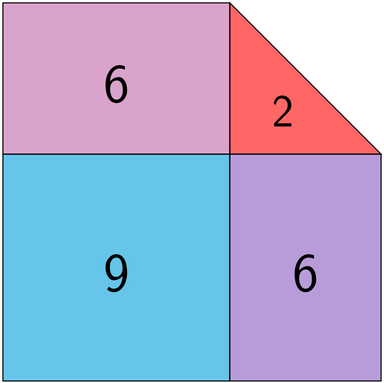
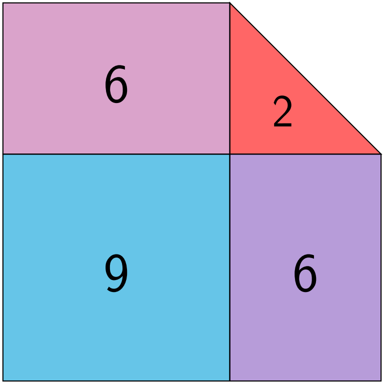
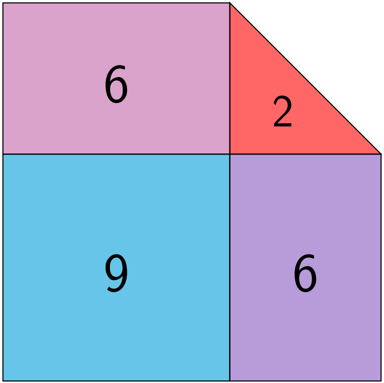


$$\mathrm{MVol}(P,P)\lambda_1^2 + \mathrm{MVol}(P,Q)\lambda_1\lambda_2 + \mathrm{MVol}(Q,P)\lambda_2\lambda_1 + \mathrm{MVol}(Q,Q)\lambda_2^2$$



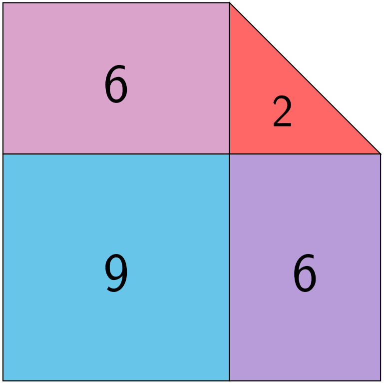
$$\mathrm{MVol}(P,P)3^2 + \mathrm{MVol}(P,Q)3\cdot 2 + \mathrm{MVol}(Q,P)2\cdot 3 + \mathrm{MVol}(Q,Q)2^2$$

Let's Do An Example



$\mathrm{MVol}(P, P)9 + \mathrm{MVol}(P, Q)6 + \mathrm{MVol}(Q, P)6 + \mathrm{MVol}(Q, Q)4$

Let's Do An Example



$$1 \cdot 9 + 1 \cdot 6 + 1 \cdot 6 + \frac{1}{2} \cdot 4$$

Mixed Volumes Give Us Log-Concave Sequences

Theorem (Alexandrov–Fenchel Inequality [Alexandrov 1937])

For convex bodies $P, Q, K_3, \ldots, K_n$ *in* $\mathbb{R}^n$*,*

$$\mathrm{MVol}(P, Q, K_3, \ldots, K_n)^2 \geq \mathrm{MVol}(P, P, K_3, \ldots, K_n)\,\mathrm{MVol}(Q, Q, K_3, \ldots, K_n).$$

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Corollary

For any convex bodies $P, Q \subseteq \mathbb{R}^n$, the sequence

$$\{\mathrm{MVol}(\underbrace{P, \ldots, P}_{n-k}, \underbrace{Q, \ldots, Q}_{k})\}_{k=0}^{n}$$

is log-concave.

We're Big Fans of Fans

Definition

A *fan* is a polyhedral complex where every element is a cone.

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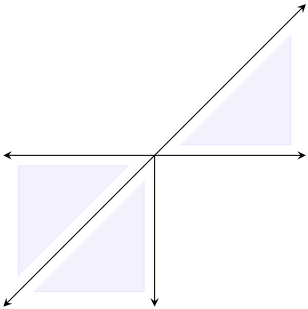
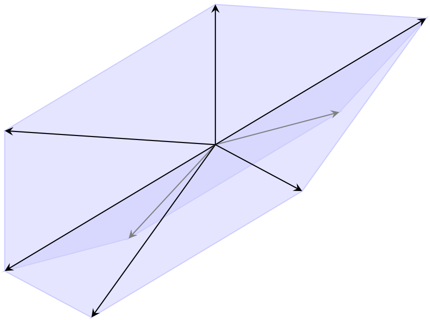




Definition (Pure)

If every maximal cone in a fan $\boldsymbol{\Sigma}$ is the same dimension, we say the fan is *pure*. We say $\boldsymbol{\Sigma}$ is a d-fan when it is pure of dimension d.

Definition (Simplicial)

A fan $\boldsymbol{\Sigma}$ is *simplicial* if for every cone $\sigma \in \boldsymbol{\Sigma}$,

$$\dim(\sigma) = |\sigma(1)|.$$

That is to say, the dimension of the cone is the same as the number of rays that generate the cone.

Definition (Tropical Fan)

Let $\mathbf{\Sigma}$ be a marked, simplicial d-fan. Given a weight function

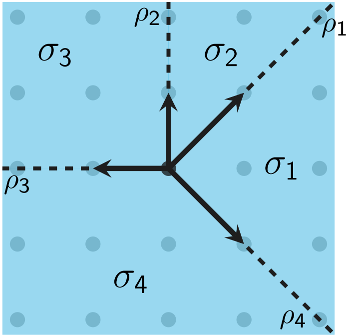
$$\omega : \mathbf{\Sigma}(d) \to \mathbb{R}_{>0},$$

we say the pair $(\mathbf{\Sigma}, \omega)$ is a *tropical fan* if for every $\tau \in \mathbf{\Sigma}(d-1)$

$$\sum_{\substack{\sigma \in \mathbf{\Sigma}(d) \\ \tau \preceq \sigma}} \omega(\sigma) u_{\sigma \setminus \tau} \in \mathrm{span}(\tau),$$

where $u_{\sigma\setminus\tau}$ is the marked vector associated to the (only) ray in $\sigma(1) \setminus \tau(1)$.

Visualising the Balancing Condition



$$\omega(\sigma_1) = 1,$$
$$\omega(\sigma_2) = \omega(\sigma_3) = \omega(\sigma_4) = 2$$

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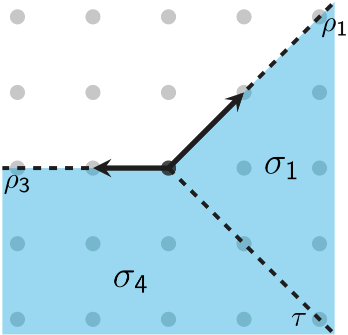

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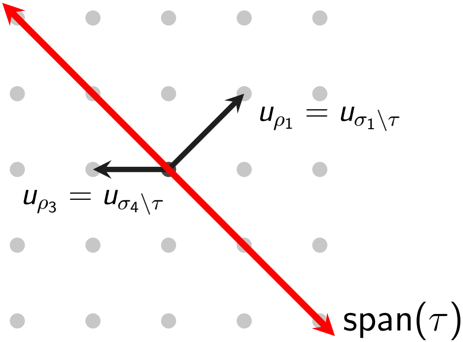

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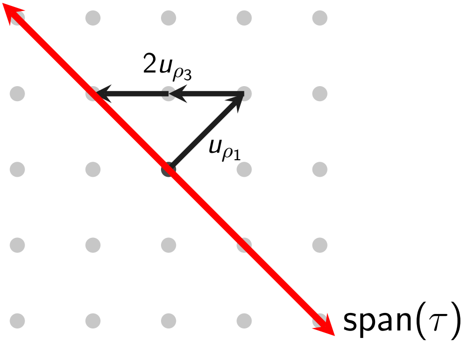

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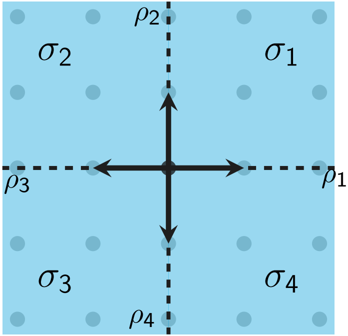
Definition (Balanced Fan)

A tropical fan $(\mathbf{\Sigma}, \omega)$ is *balanced* if the weight function ω is the constant function:

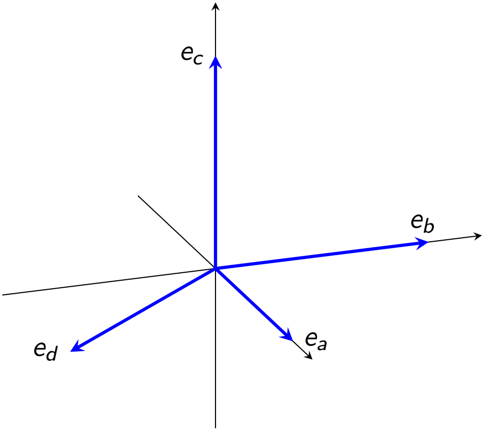
$$\omega(\sigma) = 1$$

for all $\sigma \in \mathbf{\Sigma}(d)$.



- Let $\mathcal{M}$ be a matroid with a ground set $E = \{e_0, e_1, e_2, \ldots, e_n\}$.
- Associate $e_1, e_2, \ldots, e_n$ to the standard basis vectors of R^n. Associate e_0 to the vector $[-1, \cdots, -1]$.
- Recall every flat is just a subset of ground elements. For any flat F define $e_F = \sum_{e \in F} e$.
- For every totally ordered subset of proper flats $\{F_1 \subsetneq \cdots \subsetneq F_k\} \subsetneq \mathcal{L}^*$ generate a cone using $e_{F_1}, \ldots, e_{F_k}$.

$$E = \{a, b, c, d\},\ \mathcal{L}^* = \{a, b, c, d, abd, ac, bc, cd\}.$$



Building a Bergman Fan

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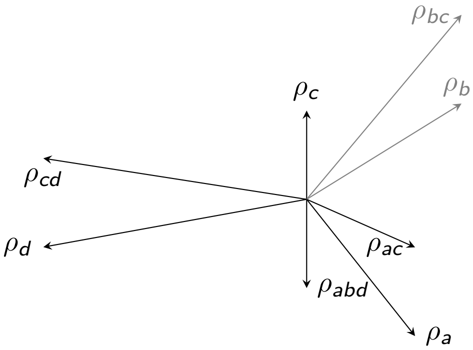

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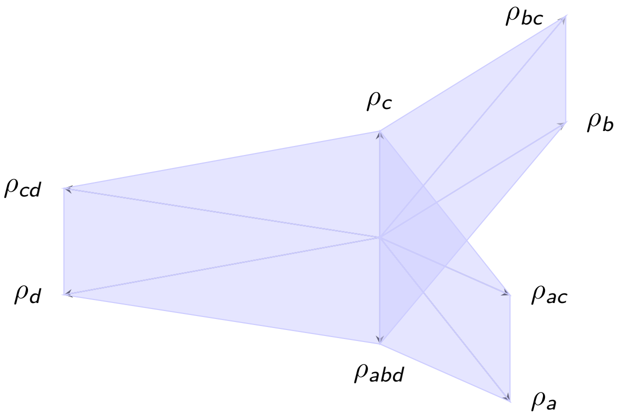

Proposition

Let $\mathcal{M}$ be a matroid of rank $r+1$, then its corresponding Bergman fan $\mathbf{\Sigma}_{\mathcal{M}}$ is a balanced, simplicial r-fan.

(Sufficiently Nice) Fans Have a Chow Ring

- There is a Chow ring associated to (nice enough) fans,

$$A^{\bullet}(\Sigma)$$

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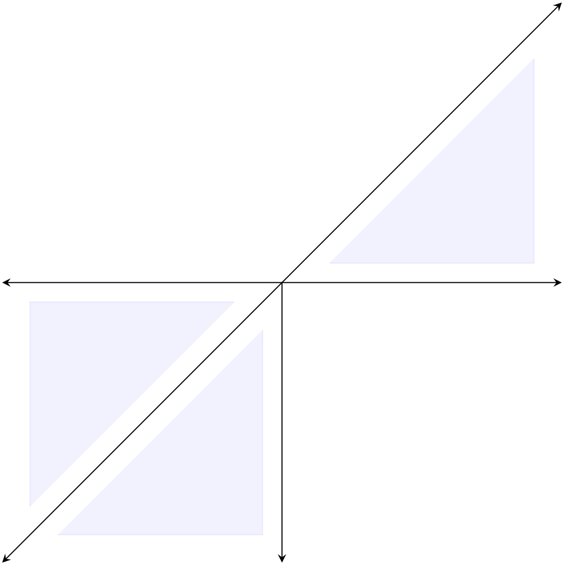
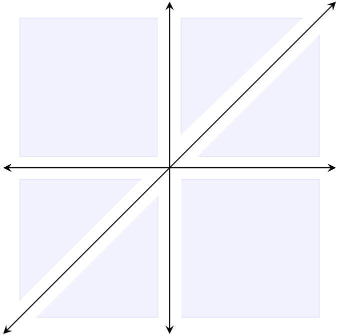
Proposition

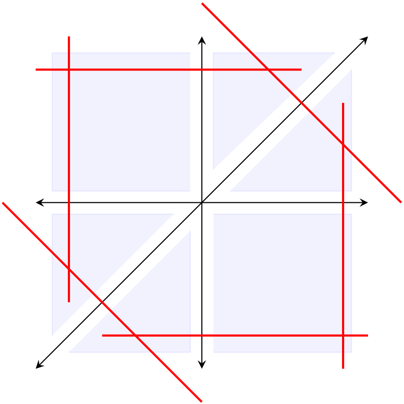
Given a matroid $\mathcal{M}$ and its Bergman fan $\mathbf{\Sigma}_\mathcal{M}$,

$$A^\bullet(\mathcal{M}) \cong A^\bullet(\mathbf{\Sigma}_\mathcal{M})$$

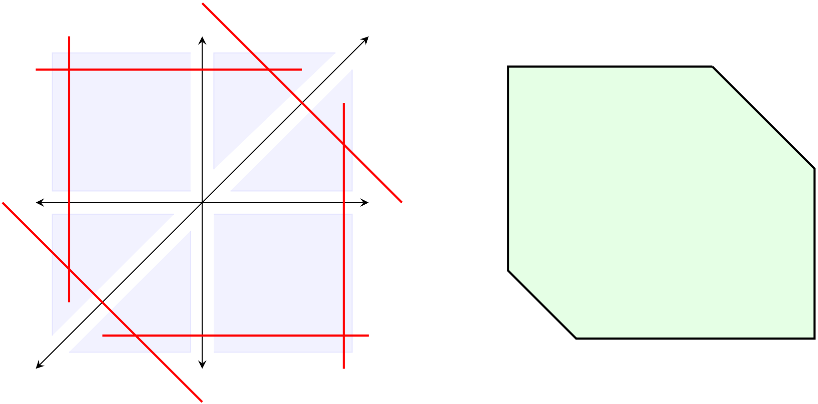
and the degree maps $\deg_\mathcal{M}$ and $\deg_{\mathbf{\Sigma}_\mathcal{M}}$ agree.







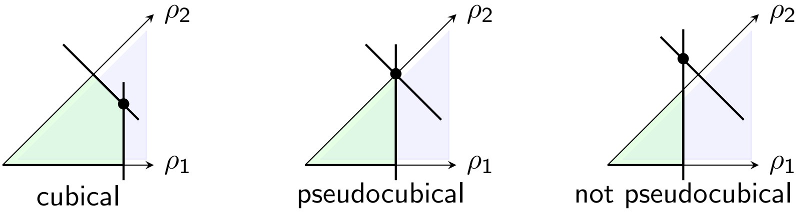
Polytopal Inspiration - The Normal Fan



Definition (Cubical Values)

For a given fan, $\Sigma \subseteq \mathbb{R}^n$, and choice of inner product, $*$, the vector $z \in \mathbb{R}^{\Sigma(1)}$ is *cubical* if for all cones $\sigma \in \Sigma$ we have

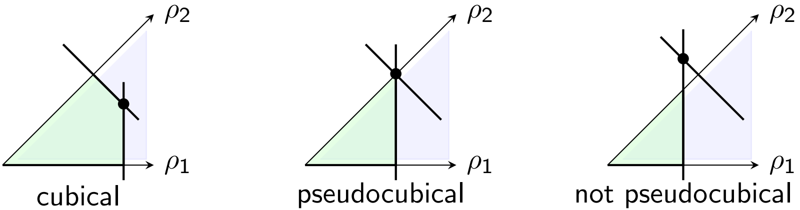
$$\sigma^\circ \cap \left\{ v \in \mathbb{R}^n \mid v * u_\rho = z_\rho \text{ for all } \rho \in \sigma(1) \right\} \neq \emptyset.$$



Definition (Psedocubical Values)

For a given fan, $\Sigma \subseteq \mathbb{R}^n$, and choice of inner product, $*$, the vector $z \in \mathbb{R}^{\Sigma(1)}$ is *pseudocubical* if for all cones $\sigma \in \Sigma$ we have

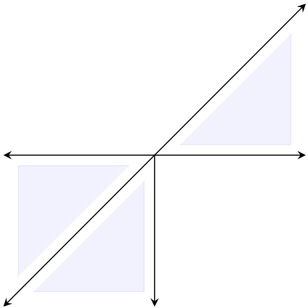
$$\sigma \cap \left\{ v \in \mathbb{R}^n \mid v * u_\rho = z_\rho \text{ for all } \rho \in \sigma(1) \right\} \neq \emptyset.$$



Definition (Normal Complex)

A normal complex is the polytopal complex resulting from intersecting a fan $\mathbf{\Sigma}$ with the halfspaces associated to the cubical value $z \in \overline{\mathsf{Cub}}(\mathbf{\Sigma}, *)$, denoted

$$C_{\mathbf{\Sigma},*}(z).$$



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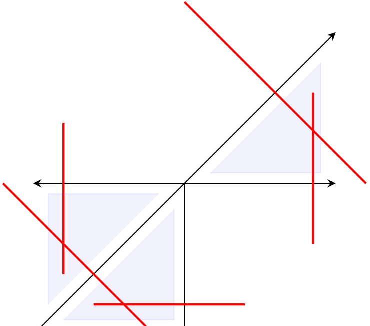

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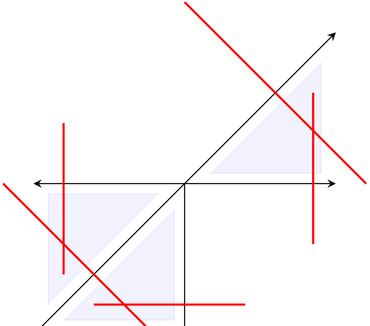
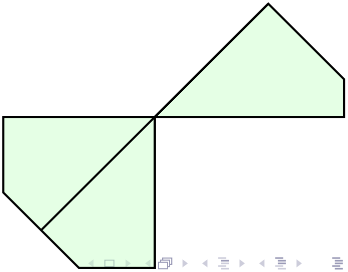
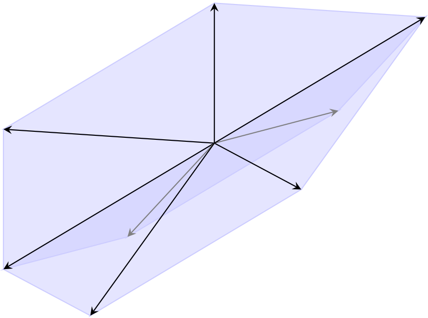
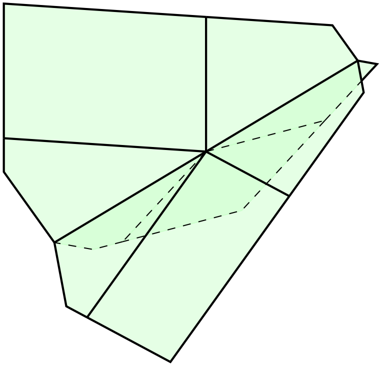




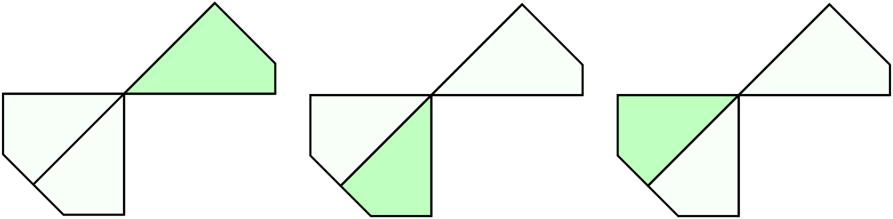
An Example a Dimension Up





The Volume of a Normal Complex

$$\mathrm{Vol}_{\boldsymbol{\Sigma},*}(z) = \sum_{\sigma \in \Sigma(d)} \mathrm{Vol}\left(P_{\sigma,*}(z)\right)$$



Big caveat that the choice of volume functions is important and we're going to just ignore that here

Theorem (Nathanson-Ross [2023])

Let $\mathbf{\Sigma} \subseteq \boldsymbol{N}$ be a balanced d-fan. Choose an inner product $* \in \mathrm{Inn}(\boldsymbol{N})$ and pseudocubical value $z \in \overline{\mathrm{Cub}}(\mathbf{\Sigma}, *)$. Define

$$D(z) = \sum_{\rho \in \mathbf{\Sigma}(1)} z_\rho x_\rho \in A^1(\mathbf{\Sigma}),$$

a divisor of the Chow ring. Then

$$\mathrm{Vol}_{\mathbf{\Sigma},*}(z) = \deg_{\mathbf{\Sigma}} \left(D(z)^d \right).$$

NB: Only when the volume is properly normalized.

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Volume is independent of choice of inner product! (It's also a degree d polynomial)

NB: Only when the volume is properly normalized.

Close But Not Quite

We have that

$$\mathsf{Vol}_{\mathbf{\Sigma},*}(z) = \deg\left(D(z)^d\right),$$

but this involves degree of a single divisor, raised to the top power.

Recall that we're interested in something like

$$\deg\left(\alpha^{r-k}\beta^k\right),$$

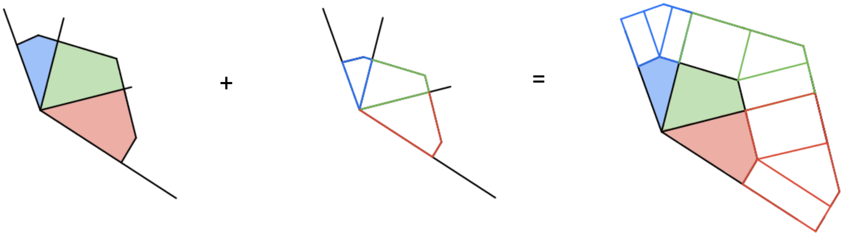
where $\alpha^{r-k}\beta^k$ is called a *mixed* degree divisor.

The Mixed Volume of a Normal Complex

Definition (Mixed Volume of Normal Complexes)

Let $\mathbf{\Sigma} \subseteq \boldsymbol{N}$ be a simplicial d-fan, $* \in \text{Inn}(\boldsymbol{N})$ be an inner product, and pseudocubical values $z_1, \ldots, z_d \in \overline{\text{Cub}}(\mathbf{\Sigma}, *)$. The mixed volume of $C_{\mathbf{\Sigma},*}(z_1), \ldots, C_{\mathbf{\Sigma},*}(z_d)$, written $\text{MVol}_{\mathbf{\Sigma},*}(z_1, \ldots, z_d)$ is given by

$$\text{MVol}_{\mathbf{\Sigma},*}(z_1, \ldots, z_d) = \sum_{\sigma \in \mathbf{\Sigma}(d)} \text{MVol}\left(P_{\sigma,*}(z_1), \ldots, P_{\sigma,*}(z_d)\right).$$



Mixed Degrees Correspond to Mixed Volumes

Proposition (Nowak-O-Ross [2023])

Let Σ be a simplicial d-fan, $*$ an inner product, and pseudocubical values $z_1, \ldots, z_n$.
The function

$$\mathrm{MVol}_{\Sigma,*}(z_1, \ldots, z_n) = \sum_{\sigma \in \Sigma(d)} \mathrm{MVol}\left(P_{\sigma,*}(z_1), \ldots, P_{\sigma,*}(z_n)\right)$$

has the following properties

1. $\mathrm{MVol}_{\Sigma,*}(z, z, \ldots, z) = \mathrm{Vol}_{\Sigma,*}(z)$,
2. $\mathrm{MVol}_{\Sigma,*}$ is symmetric in all arguments,
3. $\mathrm{MVol}_{\Sigma,*}$ is multilinear with respect to Minkowski addition in each maximal cone.

Theorem (Nowak–O–Ross)

Let $\boldsymbol{\Sigma} \subset \boldsymbol{N}$ *be a balanced d-fan. Choose an inner product* $* \in \mathrm{Inn}(\boldsymbol{N})$ *and pseudocubical values* $z_1, \ldots, z_d \in \overline{\mathrm{Cub}}(\boldsymbol{\Sigma}, *)$*. Then*

$$\mathrm{MVol}_{\boldsymbol{\Sigma},*}(z_1, \ldots, z_d) = \deg\left(D(z_1) \cdots D(z_d)\right).$$

The Alexandrov–Fenchel inequality is a classic result of *convex* geometry.

A Problem

The Alexandrov–Fenchel inequality is a classic result of *convex* geometry.

We're in a generally non-convex setting.

Sometimes These Mixed Volumes Are Nice Too

Definition (Alexandrov–Fenchel Property)

We say that $(\mathbf{\Sigma}, *)$ is *Alexandrov–Fenchel*, or just *AF*, if $\mathrm{Cub}(\mathbf{\Sigma}, *) \neq \emptyset$ and

$$\begin{aligned} &\mathrm{MVol}_{\mathbf{\Sigma},*}(z_1, z_2, z_3, \dots, z_d)^2 \\ &\quad \geq \mathrm{MVol}_{\mathbf{\Sigma},*}(z_1, z_1, z_3, \dots, z_d)\, \mathrm{MVol}_{\mathbf{\Sigma},*}(z_2, z_2, z_3 \dots, z_d) \end{aligned}$$

for all $z_1, z_2, z_3, \dots, z_d \in \mathrm{Cub}(\mathbf{\Sigma}, *)$.

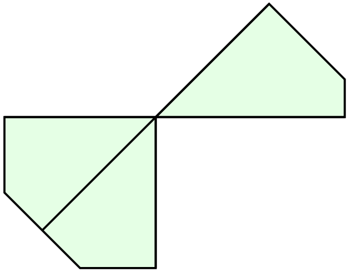
But When Are They Nice?

Theorem (Nowak–O–Ross 2023)

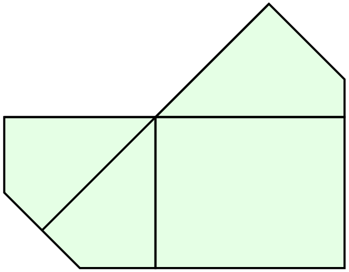
The fan Σ is AF if

1. Σ *is "pinch-free" and*
2. *the 2-dimensional faces of* Σ *are AF.*

Condition 1: "Pinch-Free" Fans

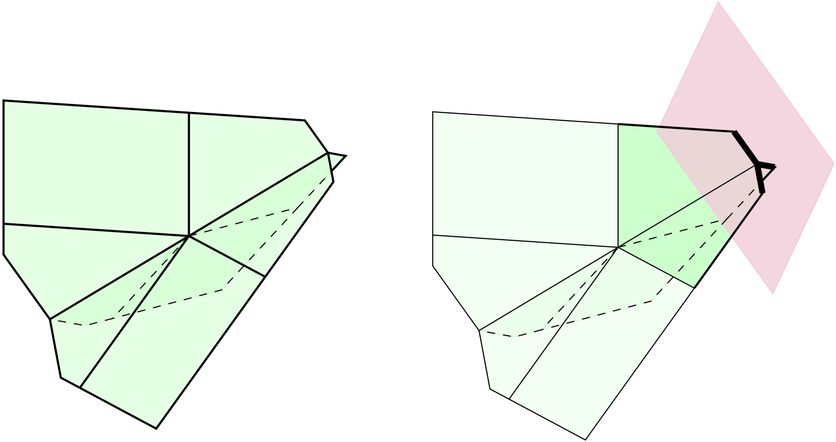


Pinched



Pinch-Free

Condition 2: 2D Faces are AF



If the Bergman fans of matroids are AF then

Some small, pesky details omitted.

If the Bergman fans of matroids are AF then

then the sequence $\{\deg(\alpha^{r-k} b^k)\}_{k=0}^{r}$ is log-concave

Some small, pesky details omitted.

The Dominos All Lined Up

If the Bergman fans of matroids are AF then

then the sequence $\{\deg(\alpha^{r-k} b^k)\}_{k=0}^{r}$ is log-concave

then the sequence of reduced coefficients $\{\overline{w}_k\}_{k=0}^{r}$ is log-concave

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then the sequence of reduced coefficients $\{\overline{w}_k\}_{k=0}^{r}$ is log-concave

then the sequence $\{w_k\}_{k=0}^{\mathrm{rk}(\mathcal{M})}$ of Whitney numbers of the first kind is log-concave, and so the Heron-Rota-Welsh conjecture is true.

Some small, pesky details omitted.

Lemma

The Bergman fan of any matroid $\mathcal{M}$ is AF.

The Big Takeaways From My Thesis

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All those pesky details work out fine.

The Big Takeaways From My Thesis

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Lemma

All those pesky details work out fine.

Theorem

The Heron–Rota–Welsh conjecture is true

Thank you!

Special thanks to Lauren Nowak and Dusty Ross