Log-Concave Sequences, Mixed Volumes, and the Normal Complex of a Fan

Some Mathematical Things Apt For An AGC Afternoon

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March 3, 2023

1 The Property

2 The Tool

3 The Object

4 What Can We Do?

Definition (Unimodal)

A sequence of numbers $a_0, a_1, \ldots, a_n$ is called unimodal if there exists some $0 \leq i \leq n$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_i \geq a_{i+1} \geq \cdots \geq a_n.$$

Definition (Log-Concave)

A sequence of numbers $a_0, a_1, \ldots, a_n$ is called logarithmically concave if for all $0 < j < n$,

$$a_j^2 \geq a_{j-1}a_{j+1}.$$

Definition (Unimodal)

Numbers only go up until the only go down.

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If our sequence is only positive numbers, then log-concave implies unimodal. Never hurts to prove the stronger condition.

The Binomial Coefficients

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

$$\binom{n}{1} \quad \binom{n}{2} \quad \cdots \quad \binom{n}{n-1} \quad \binom{n}{n}$$

How About Some Polynomials?

$1n^4 - 4n^3 + 5n^2 - 2n$

$1n^5 - 10n^4 + 35n^2 - 50n^2 + 24n$

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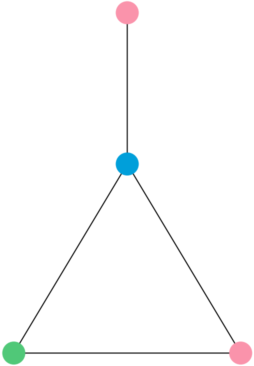

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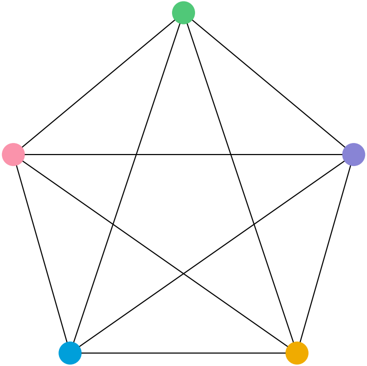

Could We Generate Some New Log-Concave Sequences?

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Sure, here's one:

1.000003, 5.06, 8.1114, 12.6732, 4.2

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Our Friend the Mixed Volume Function

Definition (The Mixed Volume Function)

The mixed volume function is a map from an ordered multiset $K_1, K_2, \ldots, K_n \subseteq \mathbb{R}^n$ of convex bodies to $\mathbb{R}_{\geq 0}$. It is uniquely characterized by the following properties:

1. $\mathrm{MVol}(K, K, \ldots, K) = \mathrm{Vol}(K)$, for any convex body K,
2. MVol is symmetric in all arguments, and
3. MVol is multilinear.

Definition (Mixed Volume – As Coefficients)

Let $K_1, K_2, \ldots, K_\ell \subseteq \mathbb{R}^n$ be convex bodies. The function

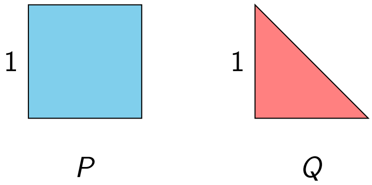
$$f(\lambda_1, \lambda_2, \ldots, \lambda_\ell) = \mathrm{Vol}(\lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_\ell K_\ell), \quad \lambda_j \geq 0$$

is symmetrically a homogeneous polynomial of degree n. It can be written as

$$f(\lambda_1, \ldots, \lambda_\ell) = \sum_{j_1, j_2, \ldots, j_n = 1}^{\ell} \mathrm{MVol}(K_{j_1}, \ldots, K_{j_n}) \lambda_{j_1} \cdots \lambda_{j_n}.$$

The coefficient associated to $\lambda_{j_1} \cdots \lambda_{j_n}$ is the mixed volume of $K_{j_1}, \ldots, K_{j_n}$.

Let's Do An Example



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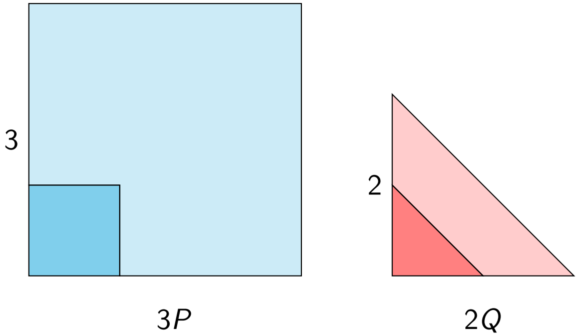

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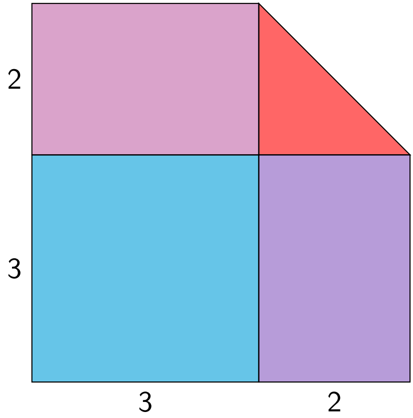
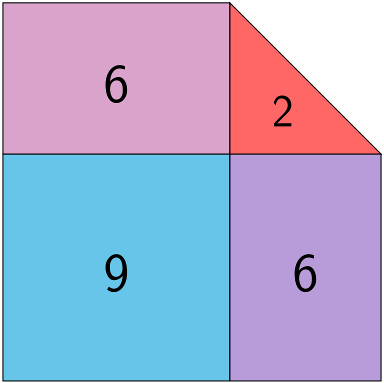


$$3P + 2Q$$



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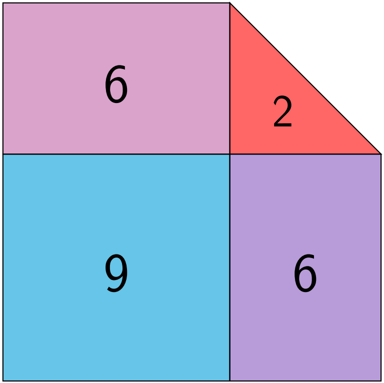
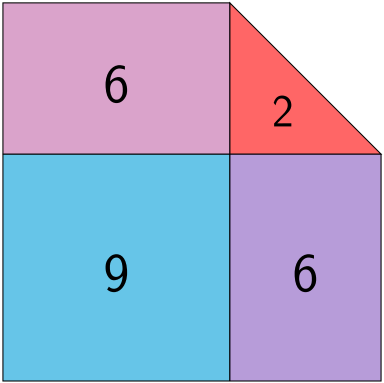


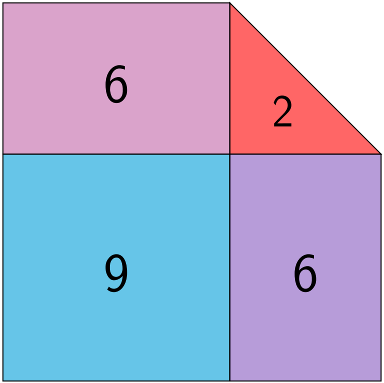
$$\mathrm{MVol}(P,P)\lambda_1^2 + \mathrm{MVol}(P,Q)\lambda_1\lambda_2 + \mathrm{MVol}(Q,P)\lambda_2\lambda_1 + \mathrm{MVol}(Q,Q)\lambda_2^2$$

Let's Do An Example



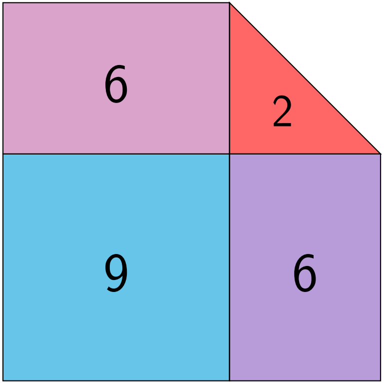
$$\mathrm{MVol}(P,P)3^2 + \mathrm{MVol}(P,Q)3\cdot 2 + \mathrm{MVol}(Q,P)2\cdot 3 + \mathrm{MVol}(Q,Q)2^2$$

Let's Do An Example



$$\mathrm{MVol}(P, P)9 + \mathrm{MVol}(P, Q)6 + \mathrm{MVol}(Q, P)6 + \mathrm{MVol}(Q, Q)4$$

Let's Do An Example



$$1 \cdot 9 + 1 \cdot 6 + 1 \cdot 6 + \frac{1}{2} \cdot 4$$

$$1,\ 1,\ 1,\ \frac{1}{2}$$

Theorem (Alexandrov–Fenchel Inequalities [Alexandrov 1937])

For convex bodies $P, Q, K_3, \ldots, K_n$ in $\mathbb{R}^n$,

$$\mathrm{MVol}(P, Q, K_3, \ldots, K_n)^2 \geq \\ \mathrm{MVol}(P, P, K_3, \ldots, K_n)\,\mathrm{MVol}(Q, Q, K_3, \ldots, K_n).$$

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Corollary

For any convex bodies $P, Q \subseteq \mathbb{R}^n$, *the sequence*

$$\Big\{ \mathrm{MVol}(\underbrace{P, \ldots, P}_{n-k}, \underbrace{Q, \ldots, Q}_{k}) \Big\}_{k=0}^{n}$$

is log-concave.

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We're Big Fans of Fans

Definition

A fan is a polyhedral complex in R^n, where every element is a cone emanating from the origin.

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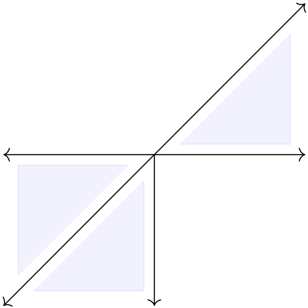

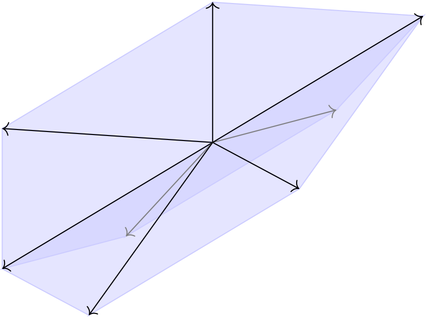

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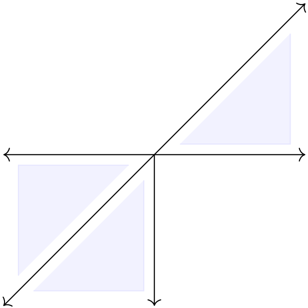
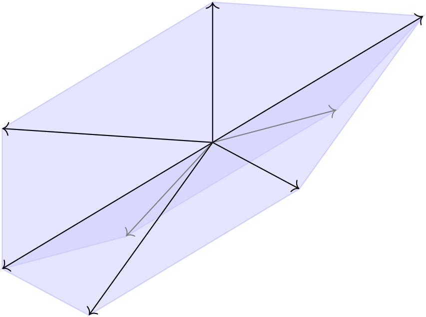




Okay *all* fans is a little broad. We are interested in nice-ish fans.

Definition (Pure)

If every maximal cone in a fan Σ is the same dimension, we say the fan is pure. We say Σ is a d-fan when it is pure of dimension d.

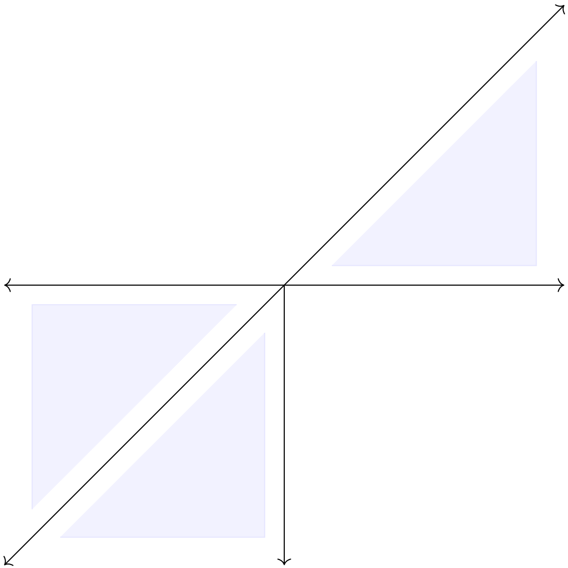
Definition (Simplicial)

A fan Σ is simplicial if for every cone $\sigma \in \Sigma$,

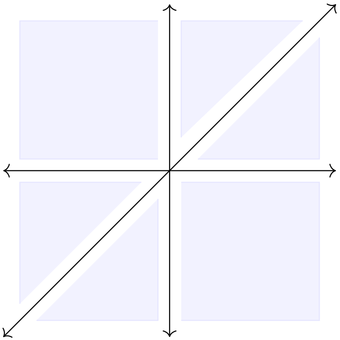
$$\dim(\sigma) = |\sigma(1)|.$$

That is to say, the dimension of the cone is the same as the number of rays that generate the cone.

Where's The Volume?



Polytopal Inspiration - The Normal Fan



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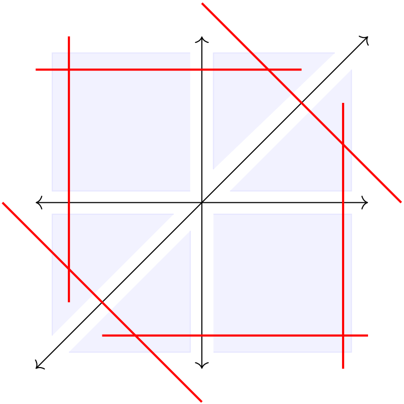

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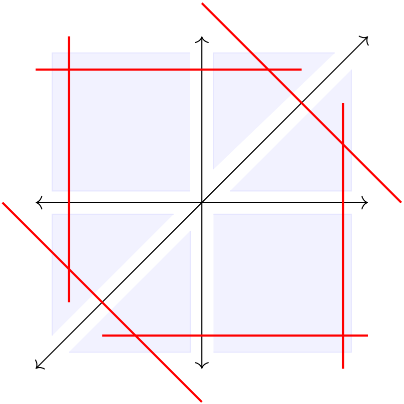
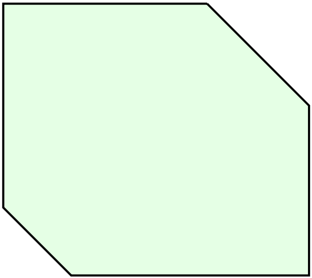
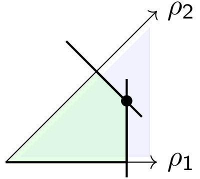




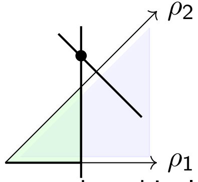
Definition (Cubical Values)

For a given fan, $\Sigma \subseteq \mathbb{R}^n$, and choice of inner product, $*$, the vector $z \in \mathbb{R}^{\Sigma(1)}$ is <u>cubical</u> if for all cones $\sigma \in \Sigma$ we have

$$\sigma^\circ \cap \left\{ v \in \mathbb{R}^n \mid v * u_\rho = z_\rho \text{ for all } \rho \in \sigma(1) \right\} \neq \emptyset.$$



cubical

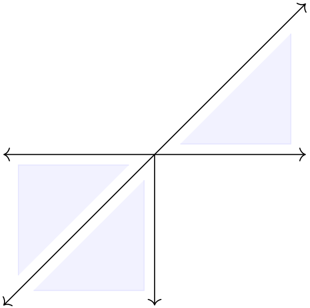


not pseudo-cubical

Definition (Normal Complex)

A normal complex is the polytopal complex resulting from intersecting a fan Σ with the halfspaces associated to the cubical value z, denoted

$$C_{\Sigma,*}(z).$$



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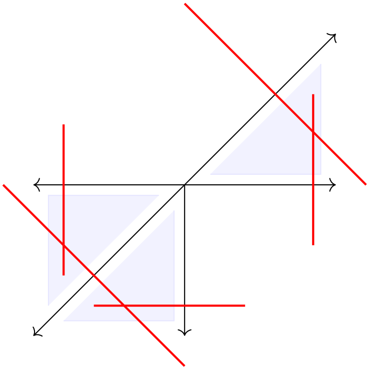

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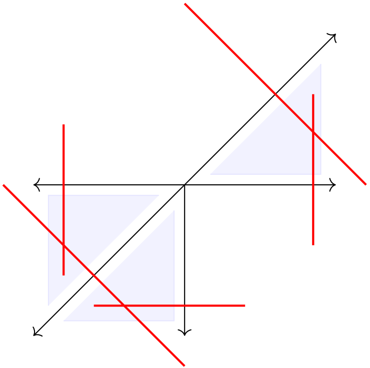
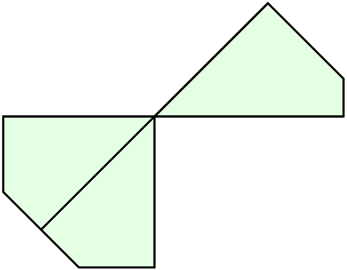
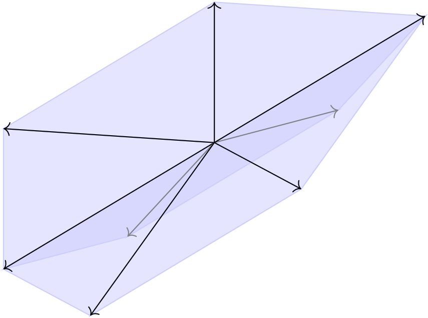
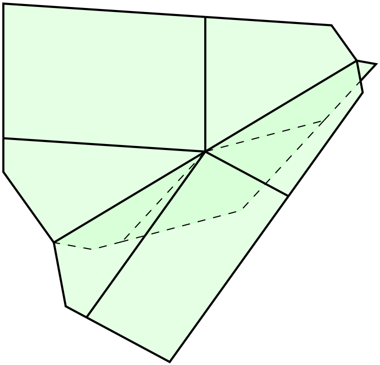


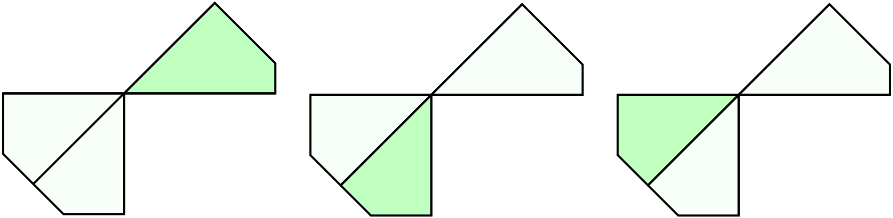


An Example a Dimension Up



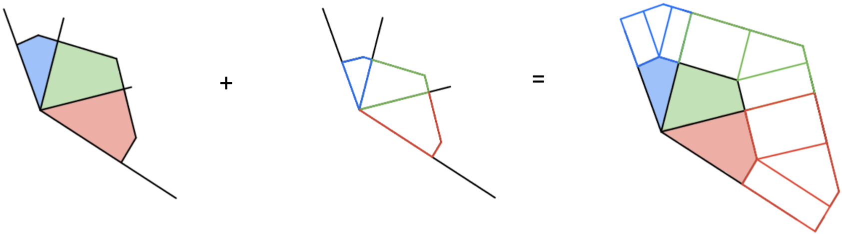


$$\overline{\text{Vol}}_{\Sigma,*}(z) = \sum_{\sigma \in \Sigma(d)} \text{Vol}\left(P_{\sigma,*}(z)\right)$$



The Mixed Volume of a Normal Complex

$$\overline{\text{MVol}}_{\Sigma,*}(z_1, \ldots, z_n) = \sum_{\sigma \in \Sigma(d)} \text{MVol}\left(P_{\sigma,*}(z_1), \ldots, P_{\sigma,*}(z_n)\right)$$



Proposition

Let $\Sigma \subset \mathbb{R}^n$ be a simplicial d-fan, $*$ an inner product on $\mathbb{R}^n$, and cubical values $z_1, \ldots, z_n$.
The function

$$\overline{\mathrm{MVol}}_{\Sigma,*}(z_1, \ldots, z_n) = \sum_{\sigma \in \Sigma(d)} \mathrm{MVol}\left(P_{\sigma,*}(z_1), \ldots, P_{\sigma,*}(z_n)\right)$$

has the following properties

1. $\overline{\mathrm{MVol}}_{\Sigma,*}(z_1, \ldots, z_n) = \overline{\mathrm{Vol}}_{\Sigma,*}(z)$,
2. $\overline{\mathrm{MVol}}_{\Sigma,*}$ is symmetric in all arguments,
3. $\overline{\mathrm{MVol}}_{\Sigma,*}$ is multilinear with respect to Minkowski addition in each maximal cone.

Are We Forgetting Something?

Our normal complexes are not convex.

While we have mixed volumes, we haven't gained anything in our pursuit of log-concave sequences.

The classic proof of the inequalities is quite intricate. Assumptions of convexity are tied all throughout.

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What we need on the geometric side is:

- Some conditions on connectivity.

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Proving Alexandrov-Fenchel

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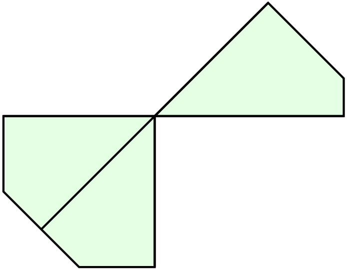
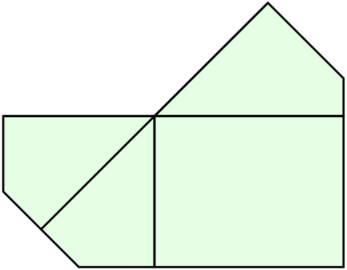
What we need on the geometric side is:

- Some conditions on connectivity.
- Being able to define volume recursively on some kind of face structure.
- The Alexandrov-Fenchel inequalities hold in dimension 2.

“Pinch-Free” Fans

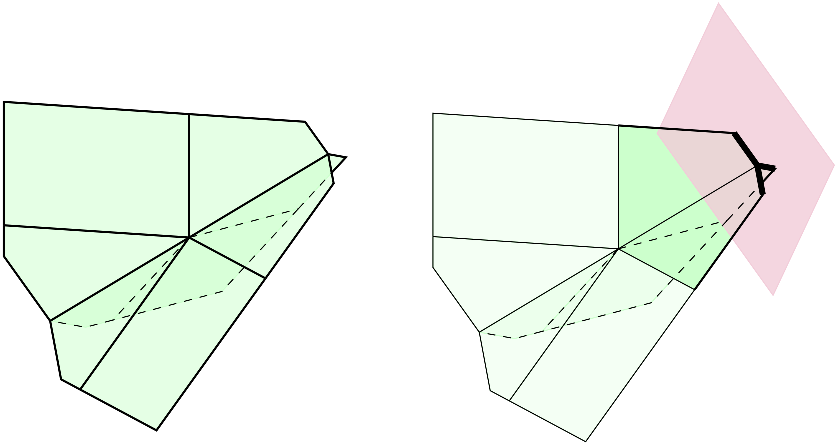


Pinched

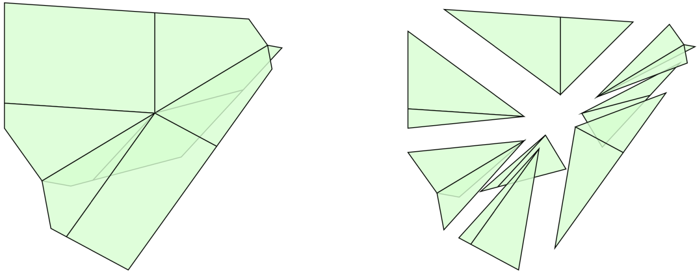


Pinch-Free

Normal Complexes Have Faces



Volume Can Be Calculated Recursively



Theorem (Nowak-O-Ross)

Let $\Sigma \subseteq \mathbb{R}^n$ be a simplicial d-fan and $*$ be an inner product of $\mathbb{R}^n$. Then $\overline{\mathrm{MVol}}_{\Sigma,*}$ obeys the Alexandrov-Fenchel inequalities for all cubical values if

- *Σ is "pinch-free", and*
- *The two-dimensional faces of $C_{\Sigma,*}(z)$ obey the Alexandrov-Fenchel inequalities.*

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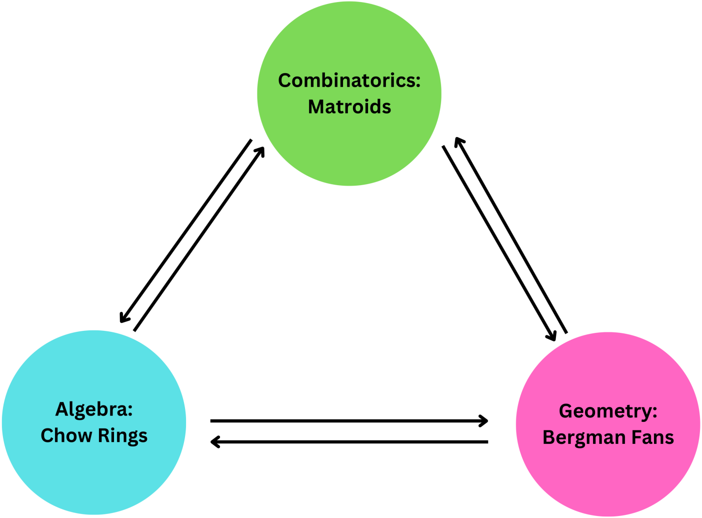
Matroids: The Mascot of AGC



Conjecture (Heron-Rota-Welsh)

The coefficients of the characteristic polynomial of a matroid, $w_0, w_1, \ldots, w_k$, form a log-concave sequence. More precisely

$$w_i^2 \geq w_{i-1} w_{i+1}$$

for $0 < i < k$.

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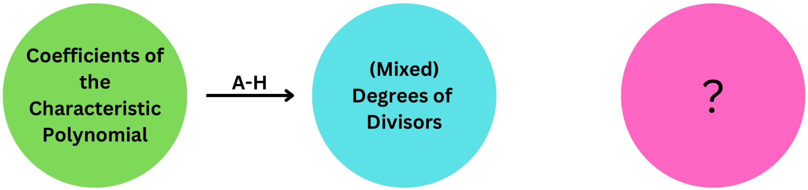
Theorem (Adiprasito-Huh-Katz 2018)

The Heron-Rota-Welsh conjecture is true.

Some Earlier Work

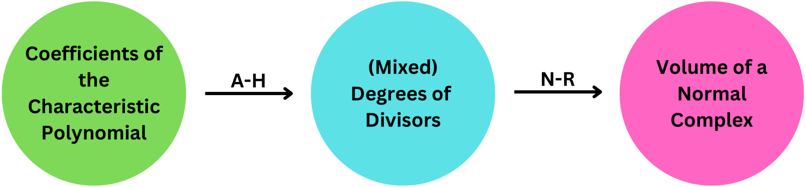
Theorem (Huh-Katz 2012)

The coefficients of the characteristic polynomial appear as mixed degrees of certain divisors (i.e. linear elements) of the Chow ring.

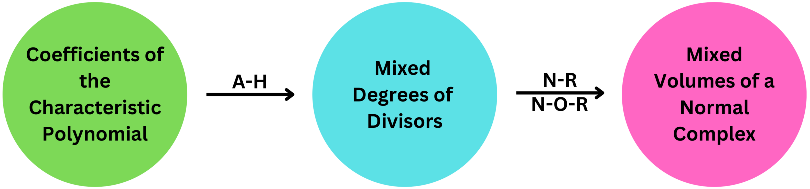


Theorem (Nathanson-Ross 2021)

If Σ is a simplicial tropical fan, then the volume of a normal complex of Σ generated by a cubical z value corresponds to the divisor whose coefficients are given by the z value.



Mixed Degrees Will Need Mixed Volumes!



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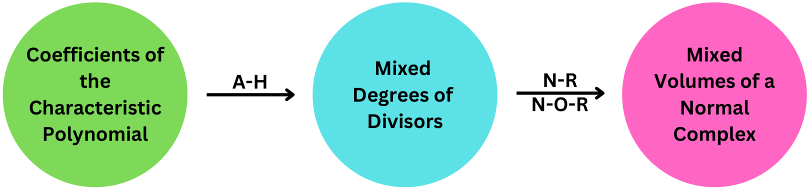

Corollary

The coefficients of the characteristic polynomial of a matroid are log-concave.

Thank you!

Special thanks to the artistic talents of Lauren Nowak and Dusty Ross.
Left to my own devices you'd all only get walls of definitions, and we'd call it a day.