The Euler Characteristic on 2-Manifolds

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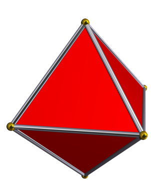
1 Introduction

1.1 Do We Need Another Topological Invariant?

Throughout the semester we have learned a number of topological properties, properties of a topological space that remain invariant under homeomorphisims. These are incredibly useful in showing two spaces are not homeomorphic, as if you can show they differ on one of these properties there cannot exist a homeomorphism between them. Sadly, however, we know now the converse is not true. Given two non-homeomorphic spaces, we could easily go down the list of every property we leaned this semester and have them match on each one.

The solution then is, naturally, find more properties to check. We wish to introduce you to one in particular, the Euler characteristic. And please do consider it very much that, an introduction. We will show its power for a small[1] class of topological spaces (specifically: compact, connected, orientable 2-manifolds), but the Euler characteristic has applications both near (e.g. compact, connected, non-orientable 2-manifolds) and far (e.g. Hoph monoids and generalized permutahedron[1]).

1.2 From Humble Beginnings



Octahedron

To get to our proof of the invariance of the Euler characteristic, we are going to go through graph theory, and to get to graph theory it will help immensely to see where Euler himself started.

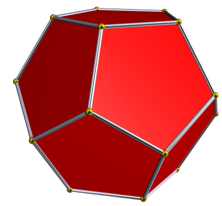
Despite the ancient Greek's deep fascination with polyhedra in general, and the platonic solids in particular, it was not until 1750 that a fundamental fact about convex polyhedra was discovered by our one and only Euler[3]. All polyhedra are formed by polygonal *faces*, which meet along lines segments called *edges*, and whose edges meet at points, i.e. *vertices*.

[1]Infinite, but a small infinite.

Theorem 1.1 (Euler's Polyhedron Formula)**.** *Given any convex polyhedron with V vertices, E edges, and F faces,*

$$V - E + F = 2.$$

From this seemingly straightforward theorem we can develop the much more general characteristic. To get there we will, in broad strokes, follow the historical route and see how we can extend this formula to graph theory. But before we can make graphs, we need something to put them on.



Dodecahedron

2 2-Manifolds a.k.a. "Surfaces"

2.1 What is a Manifold?

A n-dimensional manifold is a topological space with the property that each point has a neighborhood that is homeomorphic to the Euclidean space of dimension n. This is often called simply an n-manifold.

In this paper we only concerned with 2-manifolds, which are often called surfaces.

2.2 On Surfaces

All two-dimensional manifold are locally similar to an open disk in the plane. There are many common examples of surfaces, that we have already encountered in class. The usual sense of $\mathbb{R}^2$ with the Euclidean topology is trivially one such example. Additionally, spheres, toruses, and Möbius strips are all surfaces have we encountered in class.

We are further limiting ourselves to surfaces that are compact, connected, and orientable. We have discussed compact and connected in class, but the notion of orientability may be new.

Definition 2.1 (Orientability)**.** A surface is orientable if it is possible to make a consistent choice of a normal vector at every point on the surface.

A normal vector, as you may recall from multi-variable calculus, is the point perpendicular to the tangent plane at a point.

More intuitively, a surface is orientable if you, as a tiny person living on our manifold, can consistently point "up" in the same direction at every point. On a sphere, no matter where you go, when you point up, you'll point up away from the center of the sphere. On the surface of a torus, you would always point away from the hollow interior[2]

[2] That we have a notion of an interior is a big clue that it is orientable. Care is needed here though, as it is not difficult to think that a Klein bottle also has an "interior". You'd be wrong, the Klein bottle is non-orientable.

Now consider if we placed you were on a Möbius strip. You could point up, walk a half loop, find yourself back at the exact same point, but then point in the opposite direction. The Möbius strip is non-orientable, and so you could not consistently pick an “up”.

2.3 Some Necssary Theorems Regarding Surfaces

There are a few preliminary facts that we will both use and have to accept without proof for this paper.

2.3.1 Classification of Surfaces

Within our domain of compact, closed, orientable surfaces, we can classify these objects by the number of holes in them. The technical term for “number of holes” is genus. Thus, as sphere has no holes and is of genus 0. A torus, which has a single hole, is of genus 1, and so on. From this perspective we get the old joke, a topologist is someone who can’t tell a coffee cup from a doughnut. A coffee cup and a doughnut are genus 1, and so are hoemomorphic and thus essentially the same.

The broad statement of the theorem is as follows:

Theorem 2.1 (Classification of Surfaces)**.** *Every closed, connected, orientable surface is homeomorphic to the connected sum of g toruses for precisely one $g \in \mathbb{Z}^{\geq 0}$.*

This packs a lot of content into a deceptively simple sentence. We can alternatively state the theorem in a way that makes several important features more explicit.

Theorem 2.2 (Classification of Surfaces)**.** *Let S be a closed, connected orientable surface. Then the following hold:*

1. *The surface S is homeomorphic to a surface of genus g.*
2. *Any surface of genus g is homeomorphic to any other surface of genus g. In other words, the operation of connected sum is well defined.*
3. *If T is a surface of genus $g' \neq g$, then T is not homeomorphic to S.*

One thing we’ve not yet seen in these theorems is the connected sum.

Definition 2.2 (Connected Sum)**.** A connected sum of two m-dimensional manifolds is a manifold formed by deleting a ball inside each manifold and gluing together the resulting boundary spheres.

As it sounds, this is an operation that glues two, in our case, surfaces together such that the genus of the resulting surface is equal to the sum of the genera of the two initial surfaces.

We can restate the above informally using our earlier examples. Because there is exactly one hole in a coffee mug, and exactly one hole in a doughnut, these are both homeomorphic to the torus of genus 1. Similarly, if we were to connect the doughnut and coffee cup together, this would be equivalent to a torus of genus 2. Lastly, because the torus with two holes has genus 2 and we know $2 \neq 1$, then the torus with genus 2 is not homeomorphic to the doughnut nor the coffee mug. In this fashion we have an appropriate characterization for all compact, connected, orientable surfaces.

2.3.2 Triangulation of Surfaces

The final theorem on surfaces we need is that of triangulation. More specifically we need a theorem that tells us that our surfaces have one.

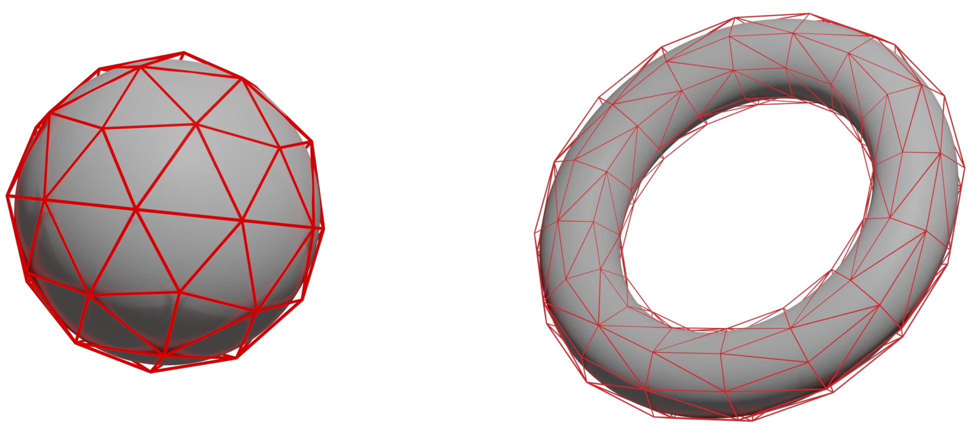
Theorem 2.3 (Rado 1925[2]). *Every compact surface is triangulable.*

This is excellent news for us, though it helps know what a triangulation of a surface is.

Definition 2.3 (Triangulation[4]). A triangulation of a topological space, X, is a finite collection of rectilinear triangle, Y, homeomorphic to X, along with the homeomorphisim $\varphi : Y \to X$.

Those rectilinear triangles are also called 2-simplices, and the whole collection Y is then a simplicial complex.

Worry not, the intuition is much less painful than the formal definition. Consider



(a) A sphere and it's potential triangulation

(b) A torus and it's potential triangulation

Figure 1: Triangulations of Simple Surfaces

the two surfaces in Figure 1. Given our intuition that things are homeomorphic if we can smoothly deform them into one another, it should not be too hard to convince yourself that the surface given by the triangles could be smoothed down into the underlying shapes. This is the heart of triangulation. Our theorem simply tells us these exist for the all surfaces we're interested in, no matter how far they stray from our nice simple examples.

If those triangulations are looking suspiciously like polyhedra, your are absolutely on the right track. The tetrahedron and octahedron are, in fact, already traingulations of the sphere! Coarse triangulations, but triangulations none the less.

Next we need to develop some of the language of graph theory, and see how we can go from polyhedra to graphs in the plane to graphs on general surfaces.

3 Graph Theory

To help us understand more about manifolds and how it relates back to the Euler's characteristic, we will introduce some graph theory.

3.1 What is a Graph?

In their most abstract form, like most mathematical objects, a graph is a set. Or, more accurately, a pair of sets.

Definition 3.1 (Graph)**.** A graph is an object consisting of two sets called its vertex set and its edge set. The vertex set is a finite, nonempty set. The edge set may be empty, but otherwise its elements are two element subsets of the vertex set[5].

Graphs are particularly nice, in that they have an obvious, if not always easily produced, representation. Graphs are dots and line in the plane. The dots are the vertices and the lines our edges.

In particular, we will be interested in a specific class of graphs called planar graphs.

Definition 3.2 (Planar Graph)**.** A graph is planar if it is isomorphic to a graph that has been drawn in a plane without edge-crossings. Otherwise a graph is nonplanar.

3.2 From Polyhedra to Graphs

Now that we have some idea of what a graph is, helps to see how Euler's formula transitioned to something about polyhedra to a statement about graphs. Cauchy, who you may remember from analysis, provided a (more) rigorous proof Euler's polyhedron formula. His proof involved projecting a polyhedron onto a graph on the plane.

Following the idea shown in Figure 2, remove a face from the polyhedron, then transport onto this face all the other vertices without changing their number, one will obtain a planar figure made up of several polygons contained in a given contour. The remaining faces could be regarded as forming a suite of polygons contained in the outline of the removed face.

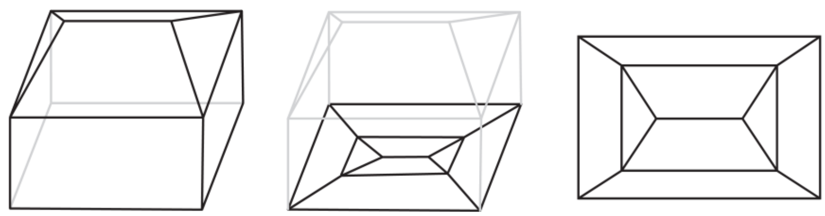


Figure 2: Cauchy projected the polyhedron into the bottom face

Cauchy didn't seem to think much of this technique beyond it's utility of in the proof[3]. But we will find this is quite powerful. A result of this technique is that any convex polyhedron has a representation as a planar graph.

3.3 Graph Theory Terminology

Lets look at some definitions of graph theory terms.

A "walk" is a finite or infinite sequence of edges which joins a sequence of vertices.

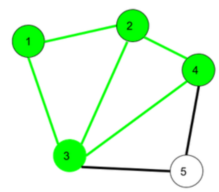


Figure 3: Walk

Here $1 \to 2 \to 3 \to 4 \to 2 \to 1 \to 3$ is a walk

A "trail" is a walk in which all edges are distinct.

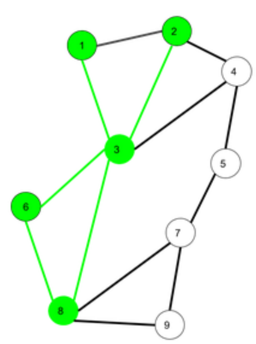


Figure 4: Trail

Here $1 \to 3 \to 8 \to 6 \to 3 \to 2$ is trail

A “path” is a trail in which all vertices (and therefore also all edges) are distinct.

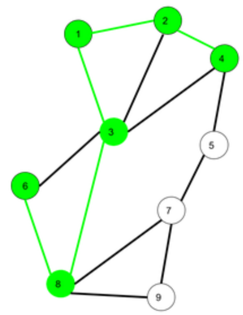


Figure 5: Path

Here $6 \to 8 \to 3 \to 1 \to 2 \to 4$ is a Path

Traversing a graph such that we do not repeat a vertex nor we repeat a edge but the starting and ending vertex must be same i.e. we can repeat starting and ending vertex only then we get a “cycle”.

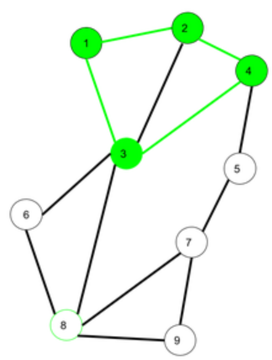


Figure 6: Cycle

Here $1 \to 2 \to 4 \to 3 \to 1$ is a cycle.

A connected acyclic graph is called a tree. In other words, a connected graph with no cycles is called a “tree”.

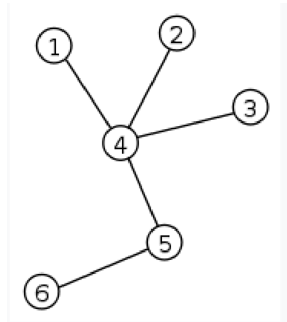


Figure 7: Tree

A "spanning tree" is a tree which includes all of the vertices of G, with a minimum possible number of edges. In general, a graph may have several spanning trees, but a graph that is not connected will not contain a spanning tree.

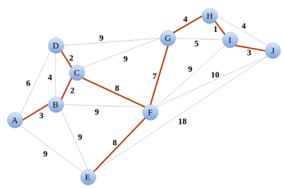


Figure 8: Spanning Tree

The "dual graph" of a planer graph G is a graph that has a vertex for each face of G. The dual graph has an edge whenever two faces of G are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge. Thus, each edge E of G has a corresponding dual edge, whose endpoints are the dual vertices corresponding to the faces on either side of e.

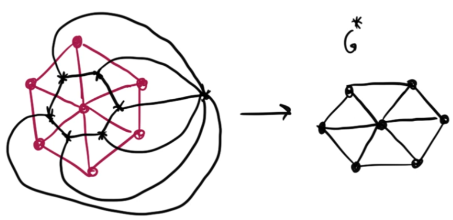


Figure 9: Dual Graph

3.4 Euler's Number

Putting all these together we are now able to see how this begins to pertain to the Euler's characteristic.

Theorem 3.1 (Euler's Number). *Given a connected, planar graph with E edges, V vertices, and F faces,*

$$V - E + F = 2$$

Proof. Notice how if we take a planer graph G, that for any tree in that graph,

$$E + 1 = V$$

If we do the same thing for the dual graph we notice that it becomes

$$E + 1 = F$$

The reason why this is relevant to us and how we can connect them together is that, so not only does the planer graph have a dual graph , any spanning tree within that graph always has a dual spanning tree in the dual graph. This key observation lets us combine these equations.

$$V + F = E + 2.$$

Now, subtract the E to the left side

$$V + F - E = 2$$

then rearrange the variables and now we get the Euler's characteristic

$$V - E + F = 2.$$

□

4 The Euler Characteristic on Orientable Surfaces

We've seen how Euler's formula went from a property of polyhedra to a property of planar graphs. There we can prove that the Euler formula holds for any planar graph, not just those we can easily get from a projection of a polyhedron. But the definition of a graph did not require any special properties of an infinite plane. If you review the definition, you'll find it works perfectly fine on anything that locally resembles a subset of $\mathbb{R}^2$. That is to say, we can put graphs on any surface.

4.1 Triangulations as Graphs and the Euler Characteristic

Of importance to us in particular are the graphs we can get from triangulations of a surfaces.

Remark. Triangulations induce a graph on the surface they're homeomorphic to.

To see this, remember that part of a triangulation is a surface itself, made of triangles with edges, vertices and faces (with the fancy name *2-simplicies*). Along with this surface, we're given a homeomorphism, φ. The image of the edges and vertices of the triangulation under φ produce the edges and vertices of a graph on the surface.

In the simple case, again imagine smoothing a triangulation of Figure 1(a) down into the sphere. The red triangles would necessarily land somewhere one the sphere, and you can then view those as your graph. The properties of homeomorphisms ensure this always works out.

With this we are ready to define the Euler characteristic[4].

Definition 4.1 (The Euler Characteristic)**.** Let M_T be a compact surface[3] with triangulation T. Let V be the number of vertices, E the number of edges, and F the number of 2-simplices in T. Then the Euler Characteristic of M_T is

$$\chi(M_T) = V - E + F$$

Our hope is that at this point the definition feels, if not obvious, then at least not out of the blue.

4.2 Invariance Up to Genus

Now that we finally have our characteristic we come to an important result. Using the Classification of Surfaces, we can tie our characteristic to the genus of a surface. Much like the surprisingly simplicity of the original formula for polyhedra, the Euler characteristic is a remarkably straightforward invariant, dependent only on the genus (and orientability, once you lose that assumption).

Our proof will use graph theory[5], while modern proofs are often firmly rooted in algebraic topology. For our proof we must first establish the characteristic of genus 0 surfaces.

Proposition 4.1. *Let X be compact, connected, orientable surface of genus 0. Then*

$$\chi(X) = 2.$$

Proof. Let X be a compact, connected, orientable surface of genus 0. Because every compact surface is triangulable, there exists some triangulation of X, T_X and a homeomorphism $\varphi : T_X \to X$. By the Classification of Surfaces, there exists some homeomorphism between X and S^2, which we will call ρ. Then their composition $\rho \circ \varphi$ is a homeomorphism from T_X to S^2. We've seen that this defines a graph on the sphere, and that all graphs that lie flat on a sphere also life flat in the plane. A planar graph has an Euler characteristic of 2, and so then must X. □

[3] Note, there is no mention of orientability.

Now we require just one more proposition for our proof.

Proposition 4.2. *For a surface of genus g with triangulation T, there exists at least one path that forms a loop in the resulting graph around each hole in the surface.*

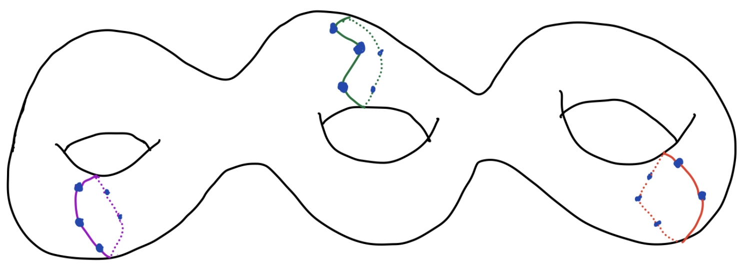


Figure 10: Loops around each hole in a surface. Only the relevant loops are drawn, the rest of the graph is left to your imagination.

This might, if you sit and play with some shapes for a bit, seem intuitively right. And while it is indeed true, the proof is non-trivial. The importance of this proposition is that is that these loops are such that that if we cut (somewhat blasphemously in a topology course) along one of the loops we take our surface down one genus.

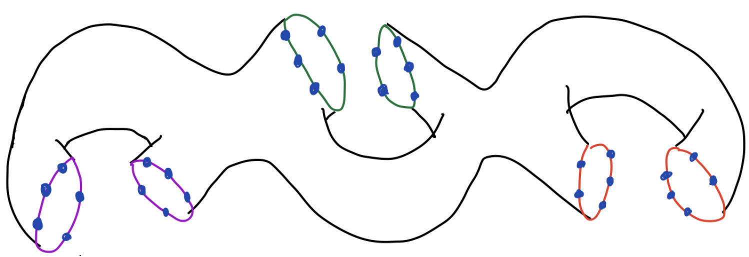


Figure 11: Our surface after cutting along the loops. With no holes left, the genus is now 0.

With this, we finally have everything we need, and can turn our attention to our main result.

Theorem 4.1 (Euler's Formula for Orientable Surfaces)**.** *If S is a compact, connected orientable surface of genus g, then*

$$\chi(S) = 2 - 2g.$$

Proof. We are working on a compact, connected, orientable surface, X of genus g. There necessarily exists some triangulation of X, T_X, which we can consider a graph on X. This graph has some number of vertices, V_X, edges, E_X, and faces F_X. The proposition above tells us there must exist at least one loop around each hole in the surface. We designate one loop around each hole, then define a new surface, Y, by splitting X along each loop, dividing each vertex and edge in the loop into 2. Each copy of the vertices and edges then define a new face. This means we have $2g$ new faces, as we made g cuts.

By the nature of its creation, Y necessarily has genus 0, and has some number of vertices V_Y, edges E_Y, faces F_Y. From the other proposition above,

$$V_Y - E_Y + F_Y = 2.$$

Now we must do some algebra. Let $x = V_Y - V_X$, the number of new vertices we added to get Y. Because each ring we cut is a loop, the number vertices and edges must be equal, so we also added x new edges. Finally, recall we added $2g$ new faces. So then

$$\begin{aligned} E_X - V_X + F_X &= (E_Y - x) - (V_Y - x) + (F_Y - 2g) \\ &= (E_Y - V_Y + F_Y) - 2g \\ &= 2 - 2g. \end{aligned}$$

With that, we are done. □

4.3 Next Steps

Our proof explicitly shied away from using more modern algebraic topology concepts. If the the Euler characteristic has piqued your interest, that's certainly the next place you may consider looking.

If you're not quite ready to wade into algebra, the immediate extensions of both the classification theorems and the Euler characteristic into non-orientable surfaces are well within your grasp.

In any case, congratulations, you have a shiny new topological invariant. Go forth and classify.

References

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[2] Tibor Radó. “Über den begriff der Riemannschen fläche”. In: Acta Litt. Sci. Szeged 2.101-121 (1925), p. 10.

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[5] Richard J Trudeau. Introduction to Graph Theory. Courier Corporation, 2013.